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Boundary noncrossings of additive Wiener fields*

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Abstract. Let $\{W_i(t),\, t \in \mathbb{R}_+\}$, $i = 1, 2$, be two Wiener processes, and let $W_3 = \{W_3(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}_+^2\}$ be a two-parameter Brownian sheet, all three processes being mutually independent. We derive upper and lower bounds for the boundary noncrossing probability $P_f = P\{W_1(t_1) + W_2(t_2) + W_3(\mathbf{t}) + f(\mathbf{t}) \leqslant u(\mathbf{t}),\ \mathbf{t} \in \mathbb{R}_+^2\}$, where $f, u : \mathbb{R}_+^2 \to \mathbb{R}$ are two general measurable functions. We further show that, for large trend functions $\gamma f > 0$, asymptotically, as $\gamma \to \infty$, $P_{\gamma f}$ is equivalent to $P_{\gamma \underline{f}}$, where $\underline{f}$ is the projection of f onto some closed convex set of the reproducing kernel Hilbert space of the field $W(\mathbf{t}) = W_1(t_1) + W_2(t_2) + W_3(\mathbf{t})$. It turns out that our approach is also applicable for the additive Brownian pillow.

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1 Introduction

Calculation of boundary noncrossing probabilities of Gaussian processes is a topic of interest for both theoretical and applied probability; see, e.g., [3, 4, 5, 6, 7, 8, 11, 14, 17, 18, 20, 22] and the references therein. Numerous applications concerned with the evaluation of boundary noncrossing probabilities relate to mathematical finance, risk theory, queueing theory, statistics, physics, among many other fields. Also, calculation of boundary noncrossing probabilities of random fields is of interest in various contexts; see, e.g., [10,12,19,21].

In this paper, we are concerned with the investigation of boundary noncrossing probabilities of an additive Wiener field, which is defined as the sum of a standard Brownian sheet and two independent Wiener processes. The choice of the model is quite natural since both the Wiener process and the Brownian sheet appear naturally as limiting processes when we consider the schemes in the domain of attraction of the central limit theorem. One one hand, these processes have continuous trajectories and independent increments, which makes our model very tractable and flexible. On the other hand, arbitrary functions defined on the positive quadrant can be decomposed uniquely into three components, two of them representing its behavior on the axes and the

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third component being zero on the axes. Hence, any trend function that we can consider here is suitable for our model.

DEFINITION 1. A Brownian sheet $\widetilde{W} = \{\widetilde{W}(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}_+^2\}$ is a Gaussian random field with zero mean and covariance function

$$\mathbf{E}\bigl\{\widetilde{W}(\mathbf{t})\widetilde{W}(\mathbf{s})\bigr\} = (s_1 \wedge t_1)(s_2 \wedge t_2).$$

By the definition, the Brownian sheet is zero on the axes, and in what follows, we shall consider its continuous modification.

Let $W_i = \{W_i(t),\, t \in \mathbb{R}_+\}$, $i = 1, 2$, be two Wiener processes, and let $W_3 = \{W_3(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}_+^2\}$ be a Brownian sheet. For two measurable functions $f, u : R_+^2 \to \mathbb{R}$, we shall investigate the boundary noncrossing probability

$$P_f = f(\mathbf{t}) + W(\mathbf{t}) \leqslant u\bigl(\mathbf{t}\bigr), \quad \mathbf{t} \in \mathbb{R}_+^2,$$

with an additive Wiener field W defined by

$$W(\mathbf{t}) = W_1(t_1) + W_2(t_2) + W_3(\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}_+^2, \tag{1.1}$$

where we assume that W_1, W_2, and W_3 are mutually independent. Clearly, the additive Wiener field W is a centered Gaussian random field with covariance function

$$\mathbf{E}\bigl\{W(\mathbf{s})W(\mathbf{t})\bigr\} = s_1 \wedge t_1 + s_2 \wedge t_2 + (s_1 \wedge t_1)(s_2 \wedge t_2), \quad \mathbf{s} = (s_1, s_2),\ \mathbf{t} = (t_1, t_2). \tag{1.2}$$

For our study, we shall modify some techniques applied for a Brownian pillow. To be more precise, we cannot apply the methods proposed for a Brownian pillow in [2, 4, 12] since they are based on the fact that it vanishes on some rectangle. Therefore, we modify essentially the methods to meet the properties of our model, and in that context, some additional conditions are introduced in our main result.

As it is commonly the case for random fields, for an additive Wiener field, explicit calculations of boundary noncrossing probabilities also are not available even for the case where both f, u are constants; see, e.g., [10]. Therefore, in our analysis, we shall derive upper and lower bounds for general measurable functions u and a function f from the reproducing kernel Hilbert space (RKHS) of W denoted by $\mathcal{H}_{2,+}$.

In order to determine $\mathcal{H}_{2,+}$, we first need to recall the corresponding RKHSs of W_1, W_2, and W_3. It is well known (see, e.g., [1]) that the RKHS of the Wiener process W_1, denoted by $\mathcal{H}_1$, is characterized as follows:

$$\mathcal{H}_1 = \left\{ h : \mathbb{R}_+ \to \mathbb{R} \;\middle|\; h(t) = \int\limits_{[0,t]} h'(s)\,\mathrm{d}s,\ h' \in L_2(\mathbb{R}_+, \lambda_1) \right\}$$

with the inner product $\langle h, g\rangle = \int_{\mathbb{R}_+} h'(s)g'(s)\,\mathrm{d}s$ and the corresponding norm $\|h\|^2 = \langle h, h\rangle$. It is also well known that the RKHS of the Brownian sheet W_3, denoted by $\mathcal{H}_2$, is characterized as follows:

$$\mathcal{H}_2 = \left\{ h : \mathbb{R}_+^2 \to \mathbb{R} \;\middle|\; h(\mathbf{t}) = \int\limits_{[0,\mathbf{t}]} h''(\mathbf{s})\,\mathrm{d}\mathbf{s},\ h'' \in L_2\bigl(\mathbb{R}_+^2, \lambda_2\bigr) \right\}$$

with the inner product $\langle h, g\rangle = \int_{\mathbb{R}_+^2} h''(\mathbf{s})g''(\mathbf{s})\,\mathrm{d}\mathbf{s}$ and the corresponding norm $\|h\|^2 = \langle h, h\rangle$. Here the symbols λ_1 and λ_2 stand for the Lebesgue measures in the $\mathbb{R}_+^1$ and $\mathbb{R}_+^2$, respectively. As shown in Lemma A.2 in Appendix, the RKHS corresponding to the covariance function of the additive Wiener field W given by (1.2) is

$$\mathcal{H}_{2,+} = \left\{ h : \mathbb{R}_+^2 \to \mathbb{R} \;\middle|\; h(\mathbf{t}) = h_1(t_1) + h_2(t_2) + h_3(\mathbf{t}),\ \text{where } h_i \in \mathcal{H}_1,\ i = 1, 2 \text{ and } h_3 \in \mathcal{H}_2 \right\} \tag{1.3}$$

equipped with the inner product

$$\langle h, g\rangle = \int\limits_{\mathbb{R}_+} h_1'(s)g_1'(s)\,\mathrm{d}s + \int\limits_{\mathbb{R}_+} h_2'(s)g_2'(s)\,\mathrm{d}s + \int\limits_{\mathbb{R}_+^2} h''(\mathbf{s})g''(\mathbf{s})\,\mathrm{d}\mathbf{s} \tag{1.4}$$

and the corresponding norm $\|h\|^2 = \langle h, h\rangle$. For simplicity, we use the same notation for the norms and inner products of $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_{2,+}$. Note that, in the case where $h \in \mathcal{H}_2 \cap C^2(\mathbb{R}^2)$, we have that $h''(u,s) = \partial^2 h(u,s)/\partial u \partial s$, and it is the motivation for the notation h''.

As in [13], a direct application of Theorem $1'$ in [15] shows that for any $f \in \mathcal{H}_{2,+}$, we have

$$|P_f - P_0| \leqslant \frac{1}{\sqrt{2\pi}}\|f\|. \tag{1.5}$$

Clearly, the above inequality provides a good bound for the approximation rate of P_f by P_0 when $\|f\|$ is small. Recall that P_0 cannot be calculated explicitly; however, it can be determined with a given accuracy by simulations. More generally, if we want to compare P_f and P_g for $g \in \mathcal{H}_{2,+}$ and $g \geqslant f$, we further obtain (by [15, Thm. $1'$]) that

$$\Phi\bigl(\alpha - \|g\|\bigr) \leqslant P_g \leqslant P_f \leqslant \Phi\bigl(\alpha + \|f\|\bigr), \tag{1.6}$$

where Φ is the distribution of an $N(0,1)$ random variable, and $\alpha = \Phi^{-1}(P_0)$ is a finite constant. When $f \leqslant 0$, we can always take $g = 0$. If $f(\mathbf{t}_0) > 0$ for some $\mathbf{t}_0$ with nonnegative components, then the last inequalities are useful when $\|f\|$ is large. Indeed, for any $g \geqslant f$, $g \in \mathcal{H}_{2,+}$ using (1.6), we obtain that

$$\ln P_{\gamma f} \geqslant \ln \Phi\bigl(\alpha - \gamma\|g\|\bigr) \geqslant -\bigl(1 + o(1)\bigr)\frac{\gamma^2}{2}\|g\|^2 \quad \text{as } \gamma \to \infty,$$

and hence,

$$\ln P_{\gamma f} \geqslant -\bigl(1 + o(1)\bigr)\frac{\gamma^2}{2}\|\underline{f}\|^2, \quad \gamma \to \infty, \tag{1.7}$$

where $\underline{f}$ (which exists and is unique) satisfies

$$\min_{g, f \in \mathcal{H}_{2,+},\ g \geqslant f} \|g\| = \|\underline{f}\| > 0. \tag{1.8}$$

In Section 2, we identify $\underline{f}$ with the projection of f on a closed convex set of $\mathcal{H}_{2,+}$, and, moreover, we show that

$$\ln P_{\gamma f} \sim \ln P_{\gamma \underline{f}} \sim -\frac{\gamma^2}{2}\|\underline{f}\|^2, \quad \gamma \to \infty. \tag{1.9}$$

Our results in this paper are of both theoretical and practical interest. Furthermore, our approach can be applied when dealing, instead of an additive Wiener sheet W, with the linear combinations of W_1, W_2, and W_3. Additionally, the techniques developed in this contribution are also applicable for evaluations of boundary noncrossing probabilities of an additive Brownian pillow, i.e., when W_1 and W_2 are independent Brownian bridges, and W_3 is a Brownian pillow. In the later case, our results are more general than those in [12].

Organization of the paper is as follows. We further continue with preliminaries followed by a section containing the main result. In Appendix, we present three technical lemmas. Lemma A.1 contains Itô's formula for the product of two fields in the plane, one of them being a Brownian sheet, and the other one of bounded variation. Lemma A.2 states that the RKHS of W is determined uniquely, whereas Lemma A.3 describes the asymptotic behavior of h'' for h from the closed convex subset of $\mathcal{H}_{2,+}$ that is used for projection.

2 Preliminaries

In this paper, bold letters are reserved for vectors, so we shall write, for instance, $\mathbf{t} = (t_1, t_2) \in \mathbb{R}_+^2$, and λ_1 and λ_2 denote the Lebesgue measures on $\mathbb{R}_+$ and $\mathbb{R}_+^2$, respectively, whereas $\mathrm{d}s$ and $\mathrm{d}\mathbf{s}$ mean integration with respect to these measures.

2.1 Expansion of one-parameter functions

The results of this subsection were formulated in a different form in, e.g., [2, 12, 14]. However, we shall introduce some modifications (rewriting, for instance, V_1), which are important for the two-parameter case. From the derivations below it will become clear how to obtain expansion of multiparameter functions of two components, one of which is the "analog of the smallest concave majorant," and the other is a negative function. Specifically, when studying the boundary crossing probabilities of the Wiener process with deterministic trend $h \in \mathcal{H}_1$, it has been shown (see [3]) that the smallest concave majorant of h solves (1.8) and determines the large deviation asymptotics of this probability. Moreover, as shown in [14], the smallest concave majorant of h, which we denote by $\underline{h}$, can be written analytically as the unique projection of h onto the closed convex set

$$V_1 = \{h \in \mathcal{H}_1 \mid h'(s) \text{ is a nonincreasing function}\},$$

i.e., $\underline{h} = \mathrm{Pr}_{V_1} h$. Here we write $\mathrm{Pr}_A h$ for the projection of h onto some closed set A also for other Hilbert spaces considered below. In what follows, for a given real-valued function φ, we denote its one-parameter increments by $\Delta_s^1 \varphi(t) = \varphi(t) - \varphi(s)$, $0 \leqslant s \leqslant t < \infty$. With this notation, we can rewrite V_1 as

$$V_1 = \left\{h \in \mathcal{H}_1 \;\middle|\; \Delta_s^1 h'(t) \leqslant 0,\ 0 \leqslant s \leqslant t < \infty\right\}.$$

Lemma 1. *Let $\widetilde{V}_1 = \{h \in \mathcal{H}_1 \mid \langle h, f\rangle \leqslant 0$ for any $f \in V_1\}$ be the polar cone of V_1, and let $h \in \mathcal{H}_1$.*

(i) *If $h \in \widetilde{V}_1$, then $h \leqslant 0$.*

(ii) *We have $\langle \mathrm{Pr}_{V_1} h, \mathrm{Pr}_{\widetilde{V}_1} h\rangle = 0$ and*

$$h = \mathrm{Pr}_{V_1} h + \mathrm{Pr}_{\widetilde{V}_1} h. \tag{2.1}$$

(iii) *If $h = h_1 + h_2$, $h_1 \in V_1$, $h_2 \in \widetilde{V}_1$, and $\langle h_1, h_2\rangle = 0$, then $h_1 = \mathrm{Pr}_{V_1} h$ and $h_2 = \mathrm{Pr}_{\widetilde{V}_1} h$.*

(iv) *The unique solution of the minimization problem $\min_{g \geqslant h,\, g \in \mathcal{H}_1} \|g\|$ is $\underline{h} = \mathrm{Pr}_{V_1} h$.*

Proof. Let $h \in \widetilde{V}_1$ and define $A = \{s \in \mathbb{R}_+ \colon\ h(s) > 0\}$. Fix $T > 0$ and consider the function v such that

$$v'(s) = \int_{[s,T]} h(u) \mathbf{1}_{\{u \in A\}}\, \mathrm{d}u \mathbf{1}_{\{s \leqslant T\}}.$$

For any $0 \leqslant s \leqslant t < \infty$, we have $\Delta_s^1 v'(t) = -\int_{[s\wedge T, t\wedge T]} h(u)\mathbf{1}_{\{u\in A\}}\,\mathrm{d}u \leqslant 0$ and

$$\begin{aligned}\int_{\mathbb{R}_+} \left|v'(s)^2\right| \mathrm{d}s &= \int_{[0,T]} \left(\int_{[s,T]} h(u)\mathbf{1}_{\{u\in A\}}\,\mathrm{d}u\right)^2 \mathrm{d}s \leqslant T^2 \int_{[0,T]} h^2(u)\,\mathrm{d}u \\ &= T^2 \int_{[0,T]} \left(\int_{[0,u]} h'(s)\,\mathrm{d}s\right)^2 \mathrm{d}u \leqslant T^4 \int_{\mathbb{R}_+} \left(h'(s)\right)^2 \mathrm{d}s < \infty.\end{aligned}$$

Consequently, $v' \in L_2(\mathbb{R}_+, \lambda_1)$, $v(s) = \int_{[0,s]} v'(u)\,\mathrm{d}u \in \mathcal{H}_1$, and $v \in V_1$. Therefore,

$$\begin{aligned} 0 \geqslant \langle h, v\rangle &= \int_{\mathbb{R}_+} h'(s)v'(s)\,\mathrm{d}s = \int_{[0,T]} h'(s) \int_{[s,T]} h(u)\mathbf{1}_{\{u\in A\}}\,\mathrm{d}u\,\mathrm{d}s \\ &= \int_{[0,T]} h(u)\mathbf{1}_{\{u\in A\}} \int_{[0,u]} h'(s)\,\mathrm{d}s\,\mathrm{d}u = \int_{[0,T]} h^2(u)\mathbf{1}_{\{u\in A\}}\,\mathrm{d}u, \end{aligned} \tag{2.2}$$

implying that $\mathbf{1}_{\{u\in A\}} = 0$ λ_1-a.e. or, in other words, $h(u) \leqslant 0$ λ_1-a.e. However, h is a continuous function, and therefore, $h(u) \leqslant 0$ for any u.

Statements (ii) and (iii) follow immediately from [14] and are valid for any Hilbert space.

(iv) Write

$$f = h + \varphi = \underline{h} + \varphi + h - \underline{h} = \underline{h} + \varphi + \mathrm{Pr}_{\widetilde{V}_1} h$$

and suppose that $f \in \mathcal{H}_1$ and $\varphi \geqslant 0$. Note that for any function $g \in V_1$, its derivative g' is nonincreasing and, therefore, g' is nonnegative and $\lim_{t\to\infty} g'(t) = 0$. Since $\varphi \geqslant 0$, it follows that, for any sequence $t_n \to \infty$, we have

$$\lim_{n\to\infty} \varphi(t_n)\underline{h}'(t_n) \geqslant 0,$$

which implies

$$\begin{aligned} \langle \underline{h}, \varphi\rangle &= \int_{\mathbb{R}_+} \underline{h}'(u)\varphi'(u)\,\mathrm{d}u = \lim_{n\to\infty} \int_{[0,t_n]} \underline{h}'(u)\varphi'(u)\,\mathrm{d}u \\ &= \lim_{n\to\infty} \Bigl(\varphi(t_n)\underline{h}'(t_n) - \int_{[0,t_n]} \varphi(u)\,\mathrm{d}\bigl(\underline{h}'(u)\bigr)\Bigr) \geqslant \lim_{n\to\infty} \Bigl(-\int_{[0,t_n]} \varphi(u)\,\mathrm{d}\bigl(\underline{h}'(u)\bigr)\Bigr) \geqslant 0. \end{aligned} \tag{2.3}$$

Consequently,

$$\begin{aligned} \|f\|^2 &= \|h+\varphi\|^2 = \|\underline{h} + \varphi + \mathrm{Pr}_{\widetilde{V}_1} h\|^2 = \|\underline{h}\|^2 + 2\langle \underline{h}, \varphi\rangle + 2\langle \underline{h}, \mathrm{Pr}_{\widetilde{V}_1} h\rangle + \|\varphi + \mathrm{Pr}_{\widetilde{V}_1} h\|^2 \\ &= \|\underline{h}\|^2 + 2\langle \underline{h}, \varphi\rangle + \|\varphi + \mathrm{Pr}_{\widetilde{V}_1} h\|^2 \geqslant \|\underline{h}\|^2, \end{aligned}$$

completing the proof. □

2.2 Expansion of two-parameter functions

For a function $\varphi : \mathbb{R}_+^2 \to \mathbb{R}$, we define

$$\Delta_{\mathbf{s}}\varphi(\mathbf{t}) = \varphi(\mathbf{t}) - \varphi(s_1, t_2) - \varphi(t_1, s_2) + \varphi(\mathbf{s}),$$

$$\Delta^1_{\mathbf{s}}\varphi(t_1, s_2) = \varphi(t_1, s_2) - \varphi(\mathbf{s}), \qquad \Delta^2_{\mathbf{s}}\varphi(s_1, t_2) = \varphi(s_1, t_2) - \varphi(\mathbf{s}).$$

In our notation, $\mathbf{s} = (s_1, s_2) \leqslant \mathbf{t} = (t_1, t_2)$ means that $s_1 \leqslant t_1$ and $s_2 \leqslant t_2$. Define the closed convex set

$$V_2 = \bigl\{h \in \mathcal{H}_2 \bigm| \Delta_{\mathbf{s}} h''(\mathbf{t}) \geqslant 0,\ \Delta^1_{\mathbf{s}} h''(t_1, s_2) \leqslant 0,\ \Delta^2_{\mathbf{s}} h''(s_1, t_2) \leqslant 0 \text{ for any } \mathbf{s} \leqslant \mathbf{t} \text{ and } \mathbf{t} \in \mathbb{R}_+^2\bigr\} \tag{2.4}$$

and let $\widetilde{V}_2$ be the polar cone of V_2, namely

$$\widetilde{V}_2 = \bigl\{h \in \mathcal{H}_2 \bigm| \langle h, v\rangle \leqslant 0 \text{ for any } v \in V_2\bigr\}.$$

We further derive an expansion for two-parameter functions. Since the results are very similar to the previous lemma, we shall prove only those statements that differ in details from Lemma 1.

Lemma 2. (i) *If $h \in \widetilde{V}_2$, then $h \leqslant 0$.*

(ii) *For any $h \in \mathcal{H}_2$, we have $\langle \mathrm{Pr}_{V_2} h, \mathrm{Pr}_{\widetilde{V}_2} h\rangle = 0$ and $h = \mathrm{Pr}_{V_2} h + \mathrm{Pr}_{\widetilde{V}_2} h$.*

(iii) *If $h = h_1 + h_2$, $h_1 \in V_2$, $h_2 \in \widetilde{V}_2$, and $\langle h_1, h_2\rangle = 0$, then $h_1 = \mathrm{Pr}_{V_2} h$ and $h_2 = \mathrm{Pr}_{\widetilde{V}_2} h$.*

(iv) *For any $h \in \mathcal{H}_2$, the unique solution of the minimization problem $\min_{g \geqslant h,\, g \in \mathcal{H}_2} \|g\|$ is $\underline{h} = \mathrm{Pr}_{V_2} h$.*

Proof. We prove only statement (i). Denote $\mathbf{T} = (T, T)$, $T > 0$, and consider the function v with

$$v''(\mathbf{s}) = \int\limits_{[\mathbf{s},\mathbf{T}]} h(\mathbf{u}) \mathbf{1}_{\{\mathbf{u} \in \mathbf{A}\}} \,\mathrm{d}\mathbf{u} \mathbf{1}_{\{\mathbf{s} \leqslant \mathbf{T}\}},$$

where $\mathbf{A} = \{\mathbf{s} \in \mathbb{R}_+^2 \mid h(\mathbf{s}) \geqslant 0\}$. Then, for any $\mathbf{0} \leqslant \mathbf{s} \leqslant \mathbf{t}$,

$$\Delta_{\mathbf{s}}^1 v''(t_1, s_2) = -\int\limits_{[\mathbf{s}\wedge\mathbf{T},\,(t_1\wedge T, T)]} h(\mathbf{u}) \mathbf{1}_{\{\mathbf{u} \in \mathbf{A}\}} \,\mathrm{d}\mathbf{u} \leqslant 0,$$

$$\Delta_{\mathbf{s}}^1 v''(s_1, t_2) = -\int\limits_{[\mathbf{s}\wedge\mathbf{T},\,(T, t_2\wedge T)]} h(\mathbf{u}) \mathbf{1}_{\{\mathbf{u} \in \mathbf{A}\}} \,\mathrm{d}\mathbf{u} \leqslant 0,$$

$$\Delta_{\mathbf{s}}^2 v''(\mathbf{t}) = \int\limits_{[\mathbf{s}\wedge\mathbf{T},\,\mathbf{t}\wedge\mathbf{T}]} h(\mathbf{u}) \mathbf{1}_{\{\mathbf{u} \in \mathbf{A}\}} \,\mathrm{d}\mathbf{u} \geqslant 0.$$

Furthermore,

$$\begin{aligned}\int\limits_{\mathbb{R}_+^2} \big|v''(\mathbf{s})^2\big| \,\mathrm{d}\mathbf{s} &= \int\limits_{[\mathbf{0},\mathbf{T}]} \Bigg(\int\limits_{[\mathbf{s},\mathbf{T}]} h(\mathbf{u}) \mathbf{1}_{\{\mathbf{u} \in \mathbf{A}\}} \,\mathrm{d}\mathbf{u} \Bigg)^2 \mathrm{d}\mathbf{s} \leqslant T^4 \int\limits_{[\mathbf{0},\mathbf{T}]} h^2(\mathbf{u}) \,\mathrm{d}\mathbf{u} \\ &= T^4 \int\limits_{[\mathbf{0},\mathbf{T}]} \Bigg(\int\limits_{[\mathbf{0},\mathbf{u}]} h''(\mathbf{s}) \,\mathrm{d}\mathbf{s} \Bigg)^2 \mathrm{d}\mathbf{u} \leqslant T^8 \int\limits_{\mathbb{R}_+^2} \big(h''(\mathbf{s})\big)^2 \,\mathrm{d}\mathbf{s} < \infty.\end{aligned}$$

Consequently,

$$v'' \in L_2\big(\mathbb{R}_+^2, \lambda_2\big), \qquad v(\mathbf{s}) = \int\limits_{[\mathbf{0},\mathbf{s}]} v''(\mathbf{u}) \,\mathrm{d}\mathbf{u} \in \mathcal{H}_2,$$

and thus, $v \in V_2$. Similarly to (2.2), we conclude that $\mathbf{1}_{\{\mathbf{u} \in \mathbf{A}\}} = 0$ λ_2-a.e. Other details follow as in the proof of Lemma 1. $\square$

Since we will work with functions f from $\mathcal{H}_{2,+}$, we need to consider the projection of such an f onto a suitable closed convex set. In the following, we shall write $f = f_1 + f_2 + f_3$, meaning that $f(\boldsymbol{t}) = f_1(t_1) + f_2(t_2) + f_3(\boldsymbol{t})$, where $f_1, f_2 \in \mathcal{H}_1$ and $f_3 \in \mathcal{H}_2$. Note in passing that this decomposition is unique for any $f \in \mathcal{H}_{2,+}$. Define the closed convex set

$$V_{2,+} = \{h = h_1 + h_2 + h_3 \in \mathcal{H}_{2,+} \mid h_1, h_2 \in V_1,\ h_3 \in V_2\}$$

and let $\widetilde{V}_{2,+}$ be the polar cone of $V_{2,+}$ given by

$$\widetilde{V}_{2,+} = \bigl\{ h \in \mathcal{H}_{2,+} \bigm| \langle h, v \rangle \leqslant 0 \text{ for any } v \in V_{2,+} \bigr\}$$

with inner product from (1.4). It follows that, for any $h = h_1 + h_2 + h_3 \in \widetilde{V}_2$, we have $h_i \leqslant 0, i = 1, 2$, and $h_3 \leqslant 0$. Furthermore, $\langle \mathrm{Pr}_{V_{2,+}} h, \mathrm{Pr}_{\widetilde{V}_{2,+}} h \rangle = 0$ and

$$h = \mathrm{Pr}_{V_{2,+}} h + \mathrm{Pr}_{\widetilde{V}_{2,+}} h. \tag{2.5}$$

Similarly to Lemma 2, we also have that if $h = f + g$ with $f \in V_{2,+}$ and $g \in \widetilde{V}_{2,+}$ such that $\langle f, g \rangle = 0$, then $f = \mathrm{Pr}_{V_{2,+}} h$ and $g = \mathrm{Pr}_{\widetilde{V}_{2,+}} h$. Moreover, the unique solution of (1.8) is

$$\underline{h} = \mathrm{Pr}_{V_{2,+}} h = \mathrm{Pr}_{V_1} h_1 + \mathrm{Pr}_{V_1} h_2 + \mathrm{Pr}_{V_2} h_3. \tag{2.6}$$

3 Main result

Consider two measurable two-parameter functions $f, u : \mathbb{R}_+^2 \to \mathbb{R}$. Suppose that $f(\mathbf{0}) = 0$ and set

$$f_1(t_1) := f(t_1, 0), \qquad f_2(t_2) := f(0, t_2), \qquad f_3(\mathbf{t}) := f(\mathbf{t}) - f(t_1, 0) - f(0, t_2);$$

then we can write $f(\mathbf{t}) = f(t_1, 0) + f(0, t_2) + (f(\mathbf{t}) - f(t_1, 0) - f(0, t_2))$. Let $f_i \in \mathcal{H}_1$, $i = 1, 2$, and $f_3 \in \mathcal{H}_2$. Recall their representations $f_i(t) = \int_{[0,t]} f_i'(s)\,\mathrm{d}s$, $f_i' \in L_2(\mathbb{R}_+, \lambda_1)$, $i = 1, 2$, and $f_3(\mathbf{t}) = \int_{[0,\mathbf{t}]} f_3''(\mathbf{s})\,\mathrm{d}\mathbf{s}$, $f_3'' \in L_2(\mathbb{R}_+^2, \lambda_2)$. We shall estimate the boundary noncrossing probability

$$P_f = f(\mathbf{t}) + W(\mathbf{t}) \leqslant u(\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}_+^2.$$

In the following, we set $\underline{f_i} = \mathrm{Pr}_{V_1} f_i$, $i = 1, 2$, $\underline{f_3} = \mathrm{Pr}_{V_2} f$, and $\underline{f} = \mathrm{Pr}_{V_{2,+}} f$ and define

$$\underline{f_{13}}(t) = \underline{f_1}'(t) - \underline{f_3}''(t, 0), \qquad \underline{f_{23}}(t) = \underline{f_2}'(t) - \underline{f_3}''(0, t).$$

Note that due to the definition of the set V_2 (see (2.4)),

$$\Delta_{\mathbf{s}} \underline{f_3}''(\mathbf{t}) \geqslant 0, \quad \Delta_{\mathbf{s}}^1 \underline{f_3}''(t_1, s_2) \leqslant 0, \quad \Delta_{\mathbf{s}}^2 \underline{f_3}''(s_1, t_2) \leqslant 0 \quad \text{for any } \mathbf{s} \leqslant \mathbf{t} \text{ and } \mathbf{t} \in \mathbb{R}_+^2.$$

In general, the choice of the set V_2 is the key point of the whole work because we can easily integrate w.r.t. $\underline{f_3}''$ both in each one-parameter direction and in the plane in Riemann–Stieltjes sense. Indeed, $\underline{f_3}''$ is decreasing in each coordinate and is increasing in the two-parameter sense. We state next our main result.

Theorem 1. *Let the following conditions hold*:

(i) *both functions $\underline{f_{13}}(t)$ and $\underline{f_{23}}(t)$ are nonincreasing in their arguments*;

(ii) *the Riemann–Stieltjes integrals $\int_{[0,x]} u(x,t)\,\mathrm{d}_t(\underline{f_3}''(x,t))$, $\int_{[0,x]} u(s,x)\,\mathrm{d}_s(\underline{f_3}''(s,x))$, $\int_{\mathbb{R}_+} u(t,0)\,\mathrm{d}\underline{f_{13}}(t)$, $\int_{\mathbb{R}_+} u(0,t)\,\mathrm{d}\underline{f_{23}}(t)$, and $\int_{\mathbb{R}_+^2} u(\mathbf{t})\,\mathrm{d}\underline{f_3}''(\mathbf{t})$ exist* (*as the integrals with respect to monotonic functions*);

(iii)
$$\lim_{t\to\infty} u(t,0)\underline{f_{13}}(t) = \lim_{t\to\infty} u(0,t)\underline{f_{23}}(t) = 0, \qquad \lim_{t_1,t_2\to\infty} u(\mathbf{t})\underline{f_3}''(\mathbf{t}) = 0, \tag{3.1}$$

$$\lim_{x\to\infty} \int\limits_{[0,x]} u(x,t)\,\mathrm{d}_t\bigl(\underline{f_3}''(x,t)\bigr) = \lim_{x\to\infty} \int\limits_{[0,x]} u(s,x)\,\mathrm{d}_s\bigl(\underline{f_3}''(s,x)\bigr) = 0. \tag{3.2}$$

Then we have

$$P_f \leqslant P_{f-\underline{f}} \exp\Bigg(-\int\limits_{\mathbb{R}_+} u(t,0)\,\mathrm{d}\underline{f_{13}}(t) - \int\limits_{\mathbb{R}_+} u(0,t)\,\mathrm{d}\underline{f_{23}}(t) + \int\limits_{\mathbb{R}_+^2} u(\mathbf{t})\,\mathrm{d}\underline{f_3}''(\mathbf{t}) - \frac{1}{2}\|\underline{f}\|^2\Bigg).$$

Remark 1. Every function $f \in \mathcal{H}_{2,+}$ starts from zero. Therefore, f cannot be constant, unless $f \equiv 0$, but this case is trivial.

Remark 2. Condition (iii) of the theorem means that asymptotically the shifts and their derivatives are negligible in comparison with the function u. It is a generalization of the corresponding conditions for a Brownian bridge and Brownian pillow that are defined on compact sets, so that the corresponding condition holds automatically.

Proof of Theorem 1. Denote by $\widetilde{\mathbf{P}}$ a probability measure that is defined by its Radon–Nikodym derivative

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\widetilde{\mathbf{P}}} = \prod_{i=1,2} \exp\Bigg(-\frac{1}{2}\|f_i\|^2 + \int\limits_{\mathbb{R}_+} f_i'(t)\,\mathrm{d}W_i^0(t)\Bigg) \exp\Bigg(-\frac{1}{2}\|f_3\|^2 + \int\limits_{\mathbb{R}_+^2} f_3''(\mathbf{t})\,\mathrm{d}W_3^0(\mathbf{t})\Bigg).$$

According to the Cameron–Martin–Girsanov theorem, $W_i^0(t) = W_i(t) + \int_{[0,t]} f_i'(s)\,\mathrm{d}s$, $i = 1,2$, are independent Wiener processes, and $W_3^0(\mathbf{t}) = W_3(\mathbf{t}) + \int_{[\mathbf{0},\mathbf{t}]} f_3''(\mathbf{s})\,\mathrm{d}\mathbf{s}$ is a Brownian sheet w.r.t. the measure $\widetilde{\mathbf{P}}$ and is independent of W_1^0 and W_2^0. Denote $\mathbf{1}_u(X) = \mathbf{1}_{\{X(\mathbf{t}) \leqslant u(\mathbf{t}),\ \mathbf{t} \in \mathbb{R}_+^2\}}$ and

$$W^0(\mathbf{t}) = W_1^0(t_1) + W_2^0(t_2) + W_3^0(\mathbf{t}).$$

Since $\|f\|^2 = \|f_1\|^2 + \|f_2\|^2 + \|f_3\|^2$, using (2.5) and (2.6), we obtain

$$\begin{aligned}
P_f &= \mathbf{E}\Bigg\{\mathbf{1}_u\Bigg(\sum_{i=1,2}\big(W_i(t) + f_i(t)\big) + W_3(\mathbf{t}) + f_3(\mathbf{t})\Bigg)\Bigg\} = \mathbf{E}_{\widetilde{P}}\Bigg(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\widetilde{\mathbf{P}}}\mathbf{1}_u\big(W^0(\mathbf{t})\big)\Bigg) \\
&= \exp\Bigg(-\frac{1}{2}\|f\|^2\Bigg)\mathbf{E}\Bigg\{\exp\Bigg(\int\limits_{\mathbb{R}_+} f_1'(t)\,\mathrm{d}W_1^0(t) + \int\limits_{\mathbb{R}_+} f_2'(t)\,\mathrm{d}W_2^0(t) + \int\limits_{\mathbb{R}_+^2} f_3''(\mathbf{t})\,\mathrm{d}W_3^0(\mathbf{t})\Bigg)\mathbf{1}_u\big(W^0(\mathbf{t})\big)\Bigg\} \\
&= \exp\Bigg(-\frac{1}{2}\|\underline{f}\|^2\Bigg)\mathbf{E}\Bigg\{\prod_{i=1,2}\exp\Bigg(-\frac{1}{2}\big\|\mathrm{Pr}_{\widetilde{V}_1} f_i\big\|^2 + \int\limits_{\mathbb{R}_+} \mathrm{Pr}_{\widetilde{V}_1} f_i'(t)\,\mathrm{d}W_i^0(t)\Bigg) \\
&\quad \times \exp\Bigg(-\frac{1}{2}\big\|\mathrm{Pr}_{\widetilde{V}_2} f_3\big\|^2 + \int\limits_{\mathbb{R}_+^2} \mathrm{Pr}_{\widetilde{V}_2} f_3''(\mathbf{t})\,\mathrm{d}W_2^0(\mathbf{t})\Bigg) \\
&\quad \times \exp\Bigg(l\sum_{i=1,2}\int\limits_{\mathbb{R}_+} \underline{f_i}'(t)\,\mathrm{d}W_i^0(t) + \int\limits_{\mathbb{R}_+^2} \underline{f_3}''(\mathbf{t})\,\mathrm{d}W_2^0(\mathbf{t})\Bigg)\mathbf{1}_u\big(W^0(\mathbf{t})\big)\Bigg\}.
\end{aligned}$$

Now we only need to rewrite

$$\sum_{i=1,2}\int\limits_{\mathbb{R}_+} \underline{f_i}'(t)\,\mathrm{d}W_i^0(t) + \int\limits_{\mathbb{R}_+^2} \underline{f_3}''(\mathbf{t})\,\mathrm{d}W_3^0(\mathbf{t}) = \sum_{i=1,2}\int\limits_{\mathbb{R}_+} \underline{f_i}'(t)\,\mathrm{d}W_i^0(t) + \int\limits_{\mathbb{R}_+^2} \underline{f_3}''(\mathbf{t})\,\mathrm{d}W^0(\mathbf{t}).$$

In order to rewrite $\int_{\mathbb{R}_+} \underline{f_1}'(t)\,\mathrm{d}W_1^0(t)$, note that, in this integral, $\mathrm{d}W_1^0(t) = \mathrm{d}_1 W_1^0(t) = \mathrm{d}_1(W^0(t,0))$; therefore, on the indicator $\mathbf{1}_u(\sum_{i=1,2} W_i^0(t) + W_3^0(\mathbf{t})) = \mathbf{1}_u(W^0(\mathbf{t}))$, under the conditions of the theorem, we have the relations

$$\begin{aligned}\int\limits_{\mathbb{R}_+} \underline{f_1}'(t)\,\mathrm{d}W_1^0(t) &= \lim_{n\to\infty} \int\limits_{[0,n]} \underline{f_1}'(t)\,\mathrm{d}W_1^0(t)\\ &= \lim_{n\to\infty}\biggl(\underline{f_1}'(n)W^0(n,0) + \int\limits_{[0,n]} W^0(t,0)\,\mathrm{d}(-\underline{f_1}')(t)\biggr).\end{aligned}\tag{3.3}$$

Similarly,

$$\int\limits_{\mathbb{R}_+} \underline{f_2}'(t)\,\mathrm{d}W_2^0(t) = \lim_{n\to\infty}\biggl(\underline{f_2}'(n)W^0(0,n) + \int\limits_{[0,n]} W^0(0,t)\,\mathrm{d}(-\underline{f_2}')(t)\biggr).\tag{3.4}$$

Further, by Lemma A.1, for $\mathbf{n} = (n_1, n_2)$,

$$\begin{aligned}\int\limits_{\mathbb{R}_+^2} \underline{f_3}''(\mathbf{t})\,\mathrm{d}W^0(\mathbf{t}) = \lim_{n_1,n_2\to\infty}\biggl(&\underline{f_3}''(\mathbf{n})W^0(\mathbf{n}) - \underline{f_3}''(n_1,0)W^0(n_1,0) - \underline{f_3}''(0,n_2)W^0(0,n_2)\\ &+ \int\limits_{[\mathbf{0},\mathbf{n}]} W^0(\mathbf{t})\,\mathrm{d}\underline{f_3}''(\mathbf{t}) + \int\limits_{[0,n_1]} W^0(s,n_2)\,\mathrm{d}_s\bigl(-\underline{f_3}''(s,n_2)\bigr)\\ &+ \int\limits_{[0,n_2]} W^0(n_1,t)\,\mathrm{d}_t\bigl(-\underline{f_3}''(n_1,t)\bigr) + \int\limits_{[0,n_1]} W^0(s,0)\,\mathrm{d}_s\bigl(\underline{f_3}''(s,0)\bigr)\\ &+ \int\limits_{[0,n_2]} W^0(0,t)\,\mathrm{d}_t\bigl(\underline{f_3}''(0,t)\bigr)\biggr).\end{aligned}\tag{3.5}$$

Combining (3.3)–(3.5) and using conditions (i)–(iii) and Lemma A.3, we conclude that all values $\underline{f_3}''(\mathbf{n})$, $\underline{f_{13}}(n) = \underline{f_1}'(n) - \underline{f_3}''(n,0)$, and $\underline{f_{23}}(n) = \underline{f_2}'(n) - \underline{f_3}''(0,n)$ are nonnegative, and therefore, we get that, on the same indicator,

$$\begin{aligned}&\sum_{i=1,2}\int\limits_{\mathbb{R}_+} \underline{f_i}'(t)\,\mathrm{d}W_i^0(t) + \int\limits_{\mathbb{R}_+^2} \underline{f_3}''(t)\,\mathrm{d}W^0(\mathbf{t})\\ &\quad\leqslant \lim_{n_1,n_2\to\infty}\biggl(\underline{f_3}''(\mathbf{n})u(\mathbf{n}) + \underline{f_{13}}(n_1)u(n_1,0) + \underline{f_{23}}(n_2)u(0,n_2)\\ &\qquad + \int\limits_{[\mathbf{0},\mathbf{n}]} u(\mathbf{t})\,\mathrm{d}\underline{f_3}''(\mathbf{t}) + \int\limits_{[0,n]} u(s,n)\,\mathrm{d}_s\bigl(-\underline{f_3}''(s,n)\bigr) + \int\limits_{[0,n_2]} u(n_1,t)\,\mathrm{d}_t\bigl(-\underline{f_3}''(n_1,t)\bigr)\\ &\qquad + \int\limits_{[0,n_1]} u(s,0)\,\mathrm{d}_s(-\underline{f_{13}})(s) + \int\limits_{[0,n_2]} u(0,t)\,\mathrm{d}_t(-\underline{f_{23}})(t)\biggr)\\ &\quad\leqslant \int\limits_{\mathbb{R}_+^2} u(\mathbf{t})\,\mathrm{d}\underline{f_3}''(\mathbf{t}) + \int\limits_{\mathbb{R}_+} u(s,0)\,\mathrm{d}_s(-\underline{f_{13}})(s) + \int\limits_{\mathbb{R}_+} u(0,t)\,\mathrm{d}_t(-\underline{f_{23}})(t).\end{aligned}\tag{3.6}$$

Further conclusions are similar to those in [2]. □

If u is bounded, then according to Lemma A.3, condition (i) is satisfied. Hence, application of the theorem to $u(s,t) = u > 0$, $s,t \geqslant 0$, in combination with (1.7), implies the following result.

Corollary 1. If $f \in \mathcal{H}_{2,+}$ is such that $f(\mathbf{t}_0) > 0$ for some $\mathbf{t}_0$ with nonnegative components, then (1.9) holds, provided that both functions $\underline{f_{13}}(t)$ and $\underline{f_{23}}(t)$ are nonincreasing in their arguments.

Remark 3. (a) Our results can be generalized to higher dimensions. We only mention that in the case of n-parameter functions, we have to define similarly all the differences $\Delta^k_{\mathbf{s}} f(\mathbf{t})$, $1 \leqslant k \leqslant n$, and the space

$$V_n = \bigl\{h \in \mathcal{H}^2_n \bigm| (-1)^k \Delta^k_{\mathbf{s}} h(\mathbf{t}) \geqslant 0 \text{ for any } \mathbf{s} \leqslant \mathbf{t},\ 1 \leqslant k \leqslant n\bigr\}.$$

(b) The case of linear combinations of W_i can be treated with some obvious modifications.

(c) Consider the additive Brownian pillow

$$B(t_1,t_2) = B_1(t_1) + B_2(t_2) + B_3(t_1,t_2), \quad t_1,t_2 \in [0,1],$$

which is constructed similarly to the additive Wiener field; here B_1 and B_2 are two independent Brownian bridges, and B_3 is a Brownian pillow independent of B_1 and B_2. The RKHSs of B, B_1, B_3 are almost the same as W, W_1, W_3 with the only differences that the corresponding functions are defined on $[0,1]^2$ or $[0,1]$ and the functions vanish on the boundaries of these intervals. The closed convex spaces V_1, V_2, and V_3 are then defined similarly as in Section 2, and thus, all the results above hold for the additive Brownian pillow by simply changing the conditions for f and u accordingly. Note that, in comparison with [12], we do not need any restrictions on $\underline{f}$. Thus, the results obtained by our approach are more general.

Appendix

Let $A \in \mathcal{H}_2$ be a two-parameter nonrandom function. If $A \in \widetilde{V}_2$, then A is nonincreasing as a function of any one-parameter variable and nondecreasing as a function of two variables. Then for the additive Wiener field $W = \{W(\mathbf{t}) = W_1(t_1) + W_2(t_2) + W_3(\mathbf{t}),\ \mathbf{t} \in \mathbb{R}^2_+\}$ and for any $\mathbf{T} = (T,T)$, there exist two integrals of the first kind (according to the classification from the papers [9, 23] and [24]), $\int_{[\mathbf{0},\mathbf{T}]} A(\mathbf{u})\,\mathrm{d}W(\mathbf{u})$, which is a standard integral of a nonrandom function with respect to a Gaussian process or, equivalently, an Itô integral since, in this case,

$$\int\limits_{[\mathbf{0},\mathbf{T}]} A(\mathbf{u})\,\mathrm{d}W(\mathbf{u}) = \int\limits_{[\mathbf{0},\mathbf{T}]} A(\mathbf{u})\,\mathrm{d}W_3(\mathbf{u}),$$

and $\int_{[\mathbf{0},\mathbf{T}]} W(\mathbf{u})\,\mathrm{d}A(\mathbf{u})$, which is a Riemann–Stieltjes integral. We argue only for the existence of the integral $\int_{[\mathbf{0},\mathbf{T}]} A(\mathbf{u})\,\mathrm{d}W(\mathbf{u})$ because the existence of the integral $\int_{[\mathbf{0},\mathbf{T}]} W(\mathbf{u})\,\mathrm{d}A(\mathbf{u})$ is evident due to the continuity of the trajectories of the Wiener field. Indeed, such a function A attains its maximal value at $\mathbf{0}$. Therefore, $\int_{[\mathbf{0},\mathbf{T}]} A^2(\mathbf{s})\,\mathrm{d}\mathbf{s} \leqslant A(\mathbf{0})T^2$, which implies that $\int_{[\mathbf{0},\mathbf{T}]} A(\mathbf{u})\,\mathrm{d}W_3(\mathbf{u})$ is correctly defined as an Itô integral. Moreover, denote the increments

$$\Delta^1_{ik,n} X = \Delta^1_{(T(i-1)/n,\,T(k-1)/n)} X\left(\frac{Ti}{n}, \frac{T(k-1)}{n}\right)$$

and

$$\Delta^2_{ik,n} X = \Delta^1_{(T(i-1)/n,\,T(k-1)/n)} X\left(\frac{T(i-1)}{n}, \frac{Tk}{n}\right),$$

where $X = A, W$. Then there exist two integrals of the second kind

$$\int\limits_{[\mathbf{0},\mathbf{T}]} \mathrm{d}_i A(\mathbf{u})\,\mathrm{d}_j W(\mathbf{u}), \quad i = 1, 2,\ j = 3 - i,$$

which are defined as the limits in probability of integral sums, where, for example,

$$\int\limits_{[\mathbf{0},\mathbf{T}]} \mathrm{d}_1 A(\mathbf{u})\,\mathrm{d}_2 W(\mathbf{u}) = \lim_{n\to\infty} \sum_{1\leqslant i,k\leqslant n} \Delta^1_{ik,n} A \Delta^2_{ik,n} W.$$

Lemma A.1. *Let $A \in \widetilde{V}_2$ be a two-parameter nonrandom function, and let $W = \{W(\mathbf{t}),\ \mathbf{t} \in \mathbb{R}^2_+\}$ be an additive Wiener field. Then for any $\mathbf{T} = (T, T)$, we have the following version of the integration-by-parts formula*:

$$\begin{aligned}
\int\limits_{[\mathbf{0},\mathbf{T}]} A(\mathbf{s})\,\mathrm{d}W(\mathbf{s}) &= A(\mathbf{T})W(\mathbf{T}) - A(T,0)W(T,0) - A(0,T)W(0,T) \\
&\quad + \int\limits_{[\mathbf{0},\mathbf{T}]} W(\mathbf{s})\,\mathrm{d}A(\mathbf{s}) + \int\limits_{[0,T]} W(s,T)\,\mathrm{d}_s\bigl(-A(s,T)\bigr) + \int\limits_{[0,T]} W(T,t)\,\mathrm{d}_t\bigl(-A(T,t)\bigr) \\
&\quad + \int\limits_{[0,T]} W_1(s)\,\mathrm{d}_s\bigl(A(s,0)\bigr) + \int\limits_{[0,T]} W_2(t)\,\mathrm{d}_s\bigl(A(0,t)\bigr).
\end{aligned}$$

Proof. The standard one-parameter Itô formula yields

$$\int\limits_{[0,T]} A(s,T)\,\mathrm{d}_s W(s,T) = A(\mathbf{T})W(\mathbf{T}) - A(0,T)W(0,T) - \int\limits_{[0,T]} W(s,T)\,\mathrm{d}_s A(s,T).$$

Using further the generalized two-parameter Itô formula (see, e.g., [16]), we obtain

$$\int\limits_{[0,T]} A(s,T)\,\mathrm{d}_s W(s,T) = \int\limits_{[0,T]} A(s,0)\,\mathrm{d}W_1(s) + \int\limits_{[\mathbf{0},\mathbf{T}]} A(\mathbf{s})\,\mathrm{d}W(\mathbf{s}) + \int\limits_{[\mathbf{0},\mathbf{T}]} \mathrm{d}_1 W(\mathbf{t})\,\mathrm{d}_2 A(\mathbf{t})$$

and, similarly,

$$\int\limits_{[0,T]} W(T,t)\,\mathrm{d}_t A(T,t) = \int\limits_{[0,T]} W(0,t)\,\mathrm{d}_t A(0,t) + \int\limits_{[\mathbf{0},\mathbf{T}]} W(\mathbf{s})\,\mathrm{d}A(\mathbf{s}) + \int\limits_{[\mathbf{0},\mathbf{T}]} \mathrm{d}_1 W(\mathbf{t})\,\mathrm{d}_2 A(\mathbf{t}).$$

From the last three equalities we immediately get that

$$\begin{aligned}
\int\limits_{[\mathbf{0},\mathbf{T}]} A(\mathbf{s})\,\mathrm{d}W(\mathbf{s}) &= \int\limits_{[0,T]} A(s,T)\,\mathrm{d}_s W(s,T) - \int\limits_{[\mathbf{0},\mathbf{T}]} \mathrm{d}_1 W(\mathbf{t})\,\mathrm{d}_2 A(\mathbf{t}) - \int\limits_{[0,T]} A(s,0)\,\mathrm{d}W_1(s) \\
&= \int\limits_{[0,T]} A(s,T)\,\mathrm{d}_s W(s,T) - \int\limits_{[0,T]} W(T,t)\,\mathrm{d}_t A(T,t) + \int\limits_{[\mathbf{0},\mathbf{T}]} W(\mathbf{s})\,\mathrm{d}A(\mathbf{s})
\end{aligned}$$

$$
\begin{aligned}
&+ \int\limits_{[0,T]} W(0,t)\,\mathrm{d}_t A(0,t) - \int\limits_{[0,T]} A(s,0)\,\mathrm{d}W_1(s) \\
&= A(\mathbf{T})W(\mathbf{T}) - A(T,0)W(T,0) - A(0,T)W(0,T) \\
&+ \int\limits_{[\mathbf{0},\mathbf{T}]} W(\mathbf{s})\,\mathrm{d}A(\mathbf{s}) + \int\limits_{[0,T]} W(s,T)\,\mathrm{d}_s\bigl(-A(s,T)\bigr) + \int\limits_{[0,T]} W(T,t)\,\mathrm{d}_t\bigl(-A(T,t)\bigr) \\
&+ \int\limits_{[0,T]} W_1(s)\,\mathrm{d}_s\bigl(A(s,0)\bigr) + \int\limits_{[0,T]} W_2(t)\,\mathrm{d}_s\bigl(A(0,t)\bigr),
\end{aligned}
$$

completing the proof. □

Lemma A.2. *The RKHS of the covariance function of the additive Wiener field W coincides with $\mathcal{H}_{2,+}$ given by* (1.3).

Proof. If the function $h : \mathbb{R}_+^2 \to \mathbb{R}$ admits the representation

$$
h(\mathbf{t}) = \sum_{i=1,2} h_i(t_i) + h_3(\mathbf{t}), \tag{A.1}
$$

where $h_i \in \mathcal{H}_1$, $i = 1, 2$, and $h_3 \in \mathcal{H}_2$, then the representation (A.1) is unique. This claim follows immediately if we put $t_i = 0, i = 1, 2$. In view of (1.2), the claim follows by [1, p. 24, Thm. 5]. □

Consider the subspace $V_1 = \{h \in \mathcal{H}_1 \mid \Delta_s^1 h'(t) \leqslant 0,\ 0 \leqslant s \leqslant t < \infty\}$. Clearly, for any $h \in V_1$, we have that $h'(t) \downarrow 0$ as $t \to \infty$. Now we establish a similar fact for the subspace

$$
V_2 = \bigl\{h \in \mathcal{H}_2 \bigm| \Delta_{\mathbf{s}} h''(\mathbf{t}) \geqslant 0,\ \Delta_{\mathbf{s}}^1 h''(t_1, s_2) \leqslant 0,\ \Delta_{\mathbf{s}}^2 h''(s_1, t_2) \leqslant 0 \text{ for any } \mathbf{s} \leqslant \mathbf{t} \text{ and } \mathbf{t} \in \mathbb{R}_+^2\bigr\}.
$$

Lemma A.3. *If $h \in V_2$ is such that $\int_{\mathbb{R}_+} (h''(s,0))^2\,\mathrm{d}s < \infty$ and $\int_{\mathbb{R}_+} (h''(0,t))^2\,\mathrm{d}t < \infty$, then $h''(s,t) \downarrow 0$ as $s \to \infty$ for any $t \in R_+$, $h''(s,t) \downarrow 0$ as $t \to \infty$ for any $s \in R_+$, and $h''(s,t) \downarrow 0$ as $s, t \to \infty$.*

Proof. Note that it suffices to establish the first claim. Since $h \in V_2$, we have $\int_{\mathbb{R}_+^2} (h''(s,t))^2\,\mathrm{d}s\,\mathrm{d}t < \infty$, implying that $\int_{\mathbb{R}_+} (h''(s,t))^2\,\mathrm{d}s < \infty$ for a.e. t. Furthermore, $h''(s,t)$ is nonincreasing in s; therefore, for such t, we have $h''(s,t) \downarrow 0$ as $s \to \infty$, and it follows from the assumption that $h''(s,0) \downarrow 0$ as $s \to \infty$. Since it is nonincreasing in t, we get such a convergence for any t, and the claim follows. □

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